Lecture 23: Lovász Local Lemma

Introduction

- Let $\mathbb{B}_1, \ldots, \mathbb{B}_n$ be indicator variables for bad events in an experiment
- Suppose that each bad event is unlike, that is, we have $\mathbb{P}[\mathbb{B}_i] \leqslant p < 1$, for all $i \in \{1, \ldots, n\}$
- Our objective is to avoid all the bad events
- Observe that if $\mathbb{P}\left[\overline{\mathbb{B}_1}, \ldots, \overline{\mathbb{B}_n}\right] > 0$ then there exists a way to avoid all the bad events simultaneously
- Suppose that all the bad events $\{\mathbb{B}_1, \ldots, \mathbb{B}_n\}$ are independent
- Then, it is easy to see that

$$\mathbb{P}\left[\overline{\mathbb{B}_1}, \ldots, \overline{\mathbb{B}_n}\right] \geqslant (1-p)^n > 0$$

- Lovász Local Lemma shall help us conclude the same even in the presence of "limited dependence" between the events

Theorem

Let $(\mathbb{B}_1, \ldots, \mathbb{B}_n)$ *be the joint distribution of bad events. For each* $\mathbb{B}_i$*, where* $i \in \{1, \ldots, n\}$*, we have* $\mathbb{P}[\mathbb{B}_i] \leqslant p$ *and each event* $\mathbb{B}_i$ *depends on at most* d *other bad events. If* $\mathrm{e}p(d+1) \leqslant 1$*, then*

$$\mathbb{P}\left[\overline{\mathbb{B}_1}, \ldots, \overline{\mathbb{B}_n}\right] \geqslant \left(1 - \frac{1}{d+1}\right)^n > 0$$

The condition is also stated sometimes as $4pd \leqslant 1$ instead of $\mathrm{e}p(d+1) \leqslant 1$.

Application: k-SAT I

- Let Φ be a k-SAT formula such that each variable occurs in at most $2^{k-2}/k$ different clauses
- **Experiment.** Let $\mathbb{X}_i$ be an independent uniform random variable that assigns the variable $\mathbb{X}_i$ a values from the set $\{\text{true}, \text{false}\}$
- **Bad Events.** For the j-th clause we have the bad event $\mathbb{B}_j$ that is the indicator variable for the event: The j-th clause is not satisfied
- **Probability of a Bad Event.** For any j, note that

$$\mathbb{P}\left[\mathbb{B}_j\right] \leqslant \frac{1}{2^k}$$

Because there is at most one assignment of the variables in the clause that makes it false. It is possible that there are no assignments that make the clause false, so the failure probability is $\leqslant 1/2^k$.

Application: k-SAT II

- **Dependence.** Note that the j-th clause has k literals. The variable associated with any literal occurs in $2^{k-2}/k$ different clauses. So, the bad event $\mathbb{B}_j$ can depend on at most $d = k \cdot (2^{k-2}/k) = 2^{k-2}$ other bad events
- **Conclusion.** Note that $4pd = 1$. So, Lovász Local Lemma implies that there exists an assignment that satisfies all the clauses in the formula simultaneously
- Intuitively, this result states that if each variable is sufficiently localized in its influence then formulas have satisfiable assignments. Note that the probability p of each bas event does not depend on the overall problem-instance size (i.e., the total number of variables)

Application: Vertex Coloring

- Let G be a graph with degree at most Δ
- **Experiment.** Let $\mathbb{X}_v$ be the random variable that represents the color of the vertex $v \in V(G)$. Let $\mathbb{X}_v$ be a color chosen uniformly (and independently) at random from the set $\{1, \ldots, C\}$
- **Bad Event.** For every edge $e \in E(G)$, we have a bad event $\mathbb{B}_e$ that is the indicator variable for both its vertices receiving identical colors
- **Probability of the Bad Event.** Note that we have $\mathbb{P}\left[\mathbb{B}_e\right] = \frac{1}{C}$.
- **Dependence.** Note that the event $\mathbb{B}_e$ does not depend on any other $\mathbb{B}_{e'}$ if the edges e and e' do not share a common vertex. So, the event $\mathbb{B}_e$ depends on at most $2(\Delta - 1)$ other bad events
- **Conclusion.** A valid coloring exists if $4pd \leqslant 1$, i.e., $C \geqslant 8(\Delta - 1)$

Application: Vertex Coloring (Bad Bound)

- Let G be a graph with degree at most Δ
- **Experiment.** Let $\mathbb{X}_v$ be the random variable that represents the color of the vertex $v \in V(G)$. Let $\mathbb{X}_v$ be a color chosen uniformly (and independently) at random from the set $\{1, \ldots, C\}$
- **Bad Event.** For each vertex $v \in V(G)$, we have a bad event $\mathbb{B}_v$ that is the indicator variable for one of v's neighbors receives the same color as v
- **Probability of the Bad Event.** Note that $\mathbb{P}\left[\mathbb{B}_v\right] \leqslant 1 - \left(1 - \frac{1}{C}\right)^{\Delta}$
- **Dependence.** Note that the event $\mathbb{B}_v$ does not depend on any other event $\mathbb{B}_{v'}$ is the sets $\{v\} \cup N(v)$ and $\{v'\} \cup N(v')$ do not intersect. So, the event $\mathbb{B}_v$ depends on at most $\Delta + \Delta(\Delta - 1) = \Delta^2$ other bad events
- **Conclusion.** A valid coloring exists if $4pd \leqslant 1$, i.e., $C \geqslant$???